Solution Sheet 1

1. ★ i) Answer: $2^m - 2$.

   If a function is **not** surjective it maps to only one of the points in B. There are two possibilities, so only two of the possible 2^m functions from A to B are not surjective.

   (ii) Answer: $3^m - 3 \times 2^m + 3$.

   If $B = \{b_1, b_2, b_3\}$ then a function from A to B is **not** surjective only if it is one of the following. It either maps onto $\{b_1, b_2\}$, onto $\{b_1, b_3\}$, onto $\{b_2, b_3\}$, onto $\{b_1\}$, onto $\{b_2\}$ or onto $\{b_3\}$. By part (i) there are $3 \times (2^m - 2) + 3 \times 1$ such maps out of a possible 3^m maps.

2. $$\text{(i) } \binom{52}{13}, \quad \text{(ii) } \binom{48}{9}, \quad \text{(iii) } \binom{39}{13}, \quad \text{(iv) } \binom{52}{13} - \binom{39}{13}.$$

3. Define
   $$f : \mathcal{P}_r(A) \to \mathcal{P}_{n-r}(A) : D \longmapsto D^c \quad \forall D \subseteq A.$$
   The function f is its own inverse and is thus a bijection. Hence
   $$|\mathcal{P}_r(A)| = |\mathcal{P}_{n-r}(A)|, \quad \text{i.e.} \quad \binom{n}{r} = \binom{n}{n-r}.$$

4. $\bigcup_{r=0}^{n} \mathcal{P}_r(A)$ is the collection of **all** subsets of A, i.e. $\mathcal{P}(A)$.

   It is a disjoint union so the cardinality of the union is the sum of the cardinalities. We know that $|\mathcal{P}(A)| = 2^n$ hence
   $$2^n = \left| \bigcup_{r=0}^{n} \mathcal{P}_r(A) \right| = \sum_{r=0}^{n} |\mathcal{P}_r(A)| = \sum_{r=0}^{n} \binom{n}{r},$$
   by definition of the binomial symbols.

5. The Binomial Theorem states that for all $x, y \in \mathbb{R}$, we have
   $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$

Choose $x = -1$ and $y = 1$ to get

$$0 = (-1+1)^n = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$$

as required.

Add $\sum_{r=0}^{n} \binom{n}{r} = 2^n$, from Question 4, and $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$ to get

$$2^n = \sum_{r=0}^{n} (1 + (-1)^r) \binom{n}{r} = \sum_{\substack{r=0 \\ r \text{ even}}}^{n} 2 \binom{n}{r}.$$

Subtract to get

$$2^n = \sum_{r=0}^{n} (1 - (-1)^r) \binom{n}{r} = \sum_{\substack{r=0 \\ r \text{ odd}}}^{n} 2 \binom{n}{r}.$$

6.

$$\begin{aligned}(4x-3y)^5 &= (4x)^5 + \binom{5}{1}(4x)^4(-3y) + \binom{5}{2}(4x)^3(-3y)^2 \\ &\quad + \binom{5}{3}(4x)^2(-3y)^3 + \binom{5}{4}(4x)(-3y)^4 + (-3y)^5 \\ &= 1024x^5 - 3840x^4y + 5760x^3y^2 - 4320x^2y^3 + 1620xy^4 - 243y^5\end{aligned}$$

7. i)

$$\begin{aligned}\sum_{r=0}^{n} \frac{3^r 5^{n-r}}{r!\,(n-r)!} &= \frac{1}{n!}\sum_{r=0}^{n} 3^r 5^{n-r} \frac{n!}{r!\,(n-r)!} = \frac{1}{n!}\sum_{r=0}^{n} 3^r 5^{n-r}\binom{n}{r} \\ &= \frac{1}{n!}(3+5)^n = \frac{8^n}{n!}.\end{aligned}$$

ii)

$$\begin{aligned}\sum_{r=0}^{n} 3^{2r} 5^{n-2r}\binom{n}{r} &= \frac{1}{5^n}\sum_{r=0}^{n} 3^{2r} 5^{2n-2r}\binom{n}{r} = \frac{1}{5^n}\sum_{r=0}^{n} 9^r 25^{n-r}\binom{n}{r} \\ &= \frac{1}{5^n}(9+25)^n = \left(\frac{34}{5}\right)^n.\end{aligned}$$

8. i)

$$\sum_{r=0}^{4} 4^r \binom{4}{r} = (1+4)^4 = 25^2, \quad \text{so } x = 25.$$

ii)

$$\sum_{r=0}^{3} 3^r \binom{3}{r} = (1+3)^3 = 4^3 = 8^2, \quad \text{so } x = 8.$$

9. The coefficient of $x^{99}y^{101}$ in $(2x+3y)^{200}$ is

$$\binom{200}{99} 2^{99} 3^{101}.$$

10. ★ Proof by induction that $n^5 - n$ is divisible by 5 for all $n \geq 1$.

(i) If $n = 1$ then $n^5 - n = 0$ which is divisible by any integer, and in particular by 5.

(ii) Assume the result true for $n = k$, so 5 divides $k^5 - k$. Thus $k^5 - k = 5\ell$ for some $\ell \in \mathbb{Z}$.

Consider the $n = k + 1$ case. By the Binomial Theorem we have

$$\begin{aligned} (k+1)^5 - (k+1) &= \left(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right) - (k+1) \\ &= \left(k^5 - k\right) + 5\left(k^4 + 2k^3 + 2k^2 + k\right) \\ &= 5\left(\ell + k^4 + 2k^3 + 2k^2 + k\right) \end{aligned}$$

because of the inductive hypothesis. Hence the right hand side is divisible by 5 as must, therefore, be the left hand side, i.e. the result holds for $n = k + 1$.

Therefore by induction $n^5 - n$ is divisible by 5 for all $n \geq 1$.